COXETER GROUPS

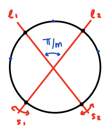
1] Geometric Reflection Groups

$\mathrm{Isom}(\mathbb{X}^n) :=$ isometry group of $\mathbb{X}^n$

1.3. Definition: A hyperplane $\mathcal{H} \in \mathbb{X}^n$ is a totally geodesic, codimension 1 submanifold of $\mathbb{X}^n$.
A hyperplane $\mathcal{H}$ separates $\mathbb{X}^n$ into two connected components, called half-spaces

For each $\mathcal{H} \subset \mathbb{X}^n$, $\exists$ a reflection $\in \mathrm{Isom}(\mathbb{X}^n)$ which a) fixes $\mathcal{H}$ and b) exchanges the associated half-spaces.

1.4 Example: finite dihedral groups

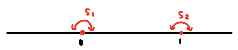


$S^1 \subset \mathbb{R}^2$

$s_1 s_2$ is a rotation by $\frac{2\pi}{m}$

$$W = \langle s_1, s_2 \rangle \overset{?}{=} D_{2m} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = e \rangle \quad \text{order } 2m$$

1.5 Example: infinite dihedral group



$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^\infty = e \rangle$$

1.7. Definition: $G \curvearrowright X$ by homeomorphisms. G_x = orbit of $x \in X$. Then a fundamental domain for $G \curvearrowright X$ is $K \subset X$ s.t.

- K is closed and connected
- $G_x \cap K \neq \emptyset \quad \forall x \in X$
- $G_x \cap K = x$ if $x \in \mathrm{int}(K)$

K is a strict fundamental domain if $G_x \cap K = x \ \forall x \in K$ K contain exactly one point from each orbit.

Dfn: Simplex $\sigma_k \subseteq \mathbb{X}^n$ $(k \le n)$ is the convex hull of $k+1$ basis vectors in $\mathbb{X}^n$. regular if all edges have the same length.

1.9. Definition: a Convex polytope $P \subseteq \mathbb{X}^n$ is a convex, compact intersection of a finite number of closed half spaces in $\mathbb{X}$ with nonempty interior

The link of a vertex v of P is $\mathrm{link}_P(v) = P \cap$ unit $(n-1)$-sphere centred at v (sphere in $T_v\mathbb{X}^n$) This is a spherical $(n-1)$-dimensional polytope. P is called simple if $\forall v \in P$, $\mathrm{link}_P(v)$ is a regular spherical simplex
Equivalently, P is called simple if each vertex is adjacent to exactly n edges.

Polytope $\neq$ Simplex.

1.11. Theorem (to prove later) $P \subseteq \mathbb{X}^n$ a simple convex polytope with $n \geq 2$. Let $\{F_i\}_{i \in I}$ be the set of codimension-1 faces of P.
Then each F_i lies in an $\mathcal{H}_i \subseteq \mathbb{X}^n$. Suppose $\forall i \neq j$ if $F_i \cap F_j \neq \emptyset$, then $\mathcal{H}_i$ and $\mathcal{H}_j$ intersect at an angle $\frac{\pi}{m_{ij}}$ $m_{ij} \geq 2 \in \mathbb{Z}$.
Set $m_{ii} = 1$ and $m_{ij} = \infty$ when $F_i \cap F_j = \emptyset$, and s_i be reflection across $\mathcal{H}_i$ in $\mathrm{Isom}(\mathbb{X}^n)$

Let W be the group generated by $\{s_i\}_{i \in I}$. Then:

1) W has the following presentation: $W = \langle s_i : (s_i s_j)^{m_{ij}} = e \quad \forall i,j \in I \rangle$
2) W is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$
3) P is a strict fundamental domain for $W \curvearrowright \mathbb{X}^n$, and the action induces a tessellation of $\mathbb{X}^n$ by copies of P.
tiling

1.13. Definition: a group W is a geometric reflection group if it is D_{2m}, D_∞ or a group from Thm 1.11. W is spherical if $\mathbb{X}^n = \mathbb{S}^n$, Euclidean if $\mathbb{X}^n = \mathbb{E}^n$, and Hyperbolic if $\mathbb{X}^n = \mathbb{H}^n$.

2. Defining Abstract reflection groups

2.1 Definition (Tits 1950s)
Let $S = \{s_i\}_{i \in I}$, I finite indexing set. A Coxeter matrix is a symmetric matrix $(S \times S)$, $M = (m_{ij})_{i,j \in I}$ such that the following hold

- $m_{ii} = 1 \quad \forall i \in I$
- $m_{ij} = m_{ji} \in \{2,3,4,\ldots\} \cup \{\infty\} \quad \forall i \neq j$.

The Coxeter group W is the group $W = \langle S \mid (s_i s_j)^{m_{ij}} = e \quad \forall i,j \in I \rangle$
and the pair (W,S) is called a Coxeter system.

Let G be a group with generating set $S \not\ni e$.
2.4. Definition: The Cayley graph of G wrt. S, $\mathrm{Cay}_S(G)$ is the graph with vertex set $= G$. It has the directed edge set $\{(g, gs) : g \in G, s \in S, s^2 \neq e\}$ and undirected edge set $\{\{g, gs\} : g \in G, s \in S, s^2 = e\}$.
All edges are labelled by coresponding $s \in S$.

$\mathrm{Cay}_S(G)$ is connected, for a cox. syst, $\mathrm{Cay}_S(W)$ is simple. → also all edges undirected.

2.6. Lemma: G acts on $\mathrm{Cay}_S(G)$ via multiplication on the left. This action preserves edge labels. if $s^2 = e$, then gsg^{-1} is the unique group element which flips the edge $\{g, gs\}$.

2.10 Definition: given a group G with generating set S of involutions, an element is a product of generators $s_1, \ldots, s_n$, $s_i \in S$, and a word is a finite sequence of generators $(s_1, \ldots, s_n)$ (care particularly about the order) $s_i \in S$. The word length of $g \in G$ wrt S is $\ell_S(g) = \min\{n \in \mathbb{N} \mid g = s_1 \ldots s_n \ \ s_i \in S\}$, and we set $\ell_S(e) = 0$. If $\ell_S(g) = n \geq 1$ and $g = s_1 \ldots s_n$, then the sequence $(s_1, \ldots, s_n)$ is called a reduced word for g.

2.11 Definition: word metric on G is $d_S(g,h) := \ell_S(g^{-1}h)$ → path metric on $\mathrm{Cay}_S(G)$.
Each edge has length 1

2.13 Definition: A pre-reflection system for G is a pair (X, R) such that:

- X is a connected, simple graph
- $G \curvearrowright X$ by graph automorphisms
- R is a subset of G and
  - a) every $r \in R$ is an involution
  - b) R is closed under conjugation : $\forall g \in G$, $r \in R$, $grg^{-1} \in R$
  - c) R generates G
  - d) $\forall \{v, w\} \in E(X)$ $\exists!$ $r \in R$ which flips $\{v, w\}$ (ie. interchanges v and w)
  - e) each $r \in R$ flips at least one edge.

For $r \in R$, let $H_r = \{$midpoints of edges flipped by $r\}$

2.15. Lemma. If (X,R) is a prereflection system for G, then G acts transitively on $V(X)$.

2.16. Lemma: let (W,S) be a Coxeter system and $R = \{wsw^{-1} \mid s \in S, w \in W\}$. Then $(\mathrm{Cay}_S(W), R)$ is a prereflection system for W.

2.17 Definition: Let (X,R) be a prereflection system for G. Then (X,R) is a reflection system if in addition it satisfies f) for each $r \in R$, $X \setminus H_r$ has exactly two components.

3. Combinatorics of Coxeter Groups

3.1 Theorem: Let W be a group generated by a set S of distinct involutions. Then the following are equivalent:

(1) (W,S) is a Coxeter system

(2) Let $X = \mathrm{Cay}_S(W)$, $R = \{ wsw^{-1} \mid s \in S, w \in W \}$. Then (X, R) is a reflection system

(3) (W,S) satisfies the 'deletion' condition

(4) (W,S) satisfies the 'exchange' condition

3.2. Definition: (W,S) satisfies the deletion condition if the following holds

(D) if $w = (s_1, \dots, s_k)$ is a word in S with $\ell_S(s_1 \dots s_k) < k$,
then $\exists$ indices $i<j$ such that $s_1 \dots s_k = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$,

3.3. Definition: (W,S) satisfies the exchange condition if the following holds

(E) If $(s_1, \dots, s_k)$ is a reduced word, then for any $s \in S$, either
$\ell_S(s s_1 \dots s_k) = k+1$, or $w = s_1 \dots s_k = s s_1 \dots \hat{s}_i \dots s_k$ for $i \in \{1, \dots, k\}$

Let $R = \{ wsw^{-1} : w \in W, s \in S \}$. $\exists !\ r \in R$ which flips each edge

$$s_1 \dots s_{j-1} \text{ ——— } s_1 \dots s_j$$

given by $r_j = s_1 \dots s_j s_{j-1} \dots s_1$. get a reflection sequence $(r_1, \dots, r_k)$ for a word $(s_1, \dots, s_k)$

3.5. Lemma: $\forall r \in R$, $\mathrm{Cay}_S(W) \setminus H_r$ has at most two connected components.

(W,S) a coxeter system. For any word $(s_1, \dots, s_k) = \underline{\Sigma}$, let $n(r, \Sigma)$ be the number of times the corresponding path crosses H_r in $\mathrm{Cay}_S(W)$.

3.6. Lemma: (i) for any word $\Sigma = (s_1, \dots, s_k)$ with $w = s_1 \dots s_k$, then for any $r \in R$, $(-1)^{n(r,\Sigma)} \in \{\pm 1\}$ depends only on $w \in W$ (not the word representation but the actual element).

(ii) $\exists$ a group homomorphism $W \longmapsto \mathrm{Sym}(R \times \{\pm 1\})$; $w \mapsto \phi_w$ s.t. $\phi_w(r, \varepsilon) = (wrw^{-1}, (-1)^{n(r,\Sigma)} \varepsilon)$, where Σ is any word representing w.

3.7 Definition: Let W be gen by a set of distinct involutions S and $s \neq t \in S$ st order of st, m_{st}, is finite. A *braid move* on a word in S swaps a subword $(s,t,s,t,\dots)$ of length m_{st} with a subword $(t,s,t,s\dots)$ of length m_{st}.

Theorem 3.10 (Tits) Suppose W a group gen by a set S of distinct involutions, (W,S) satisfies (E). Then

(1) a word $(s_1, \dots, s_k)$ is reduced $\Leftrightarrow$ it cannot be shortened by a sequence of (i) deleting a subword (s,s) $s \in S$, or
(ii) a braid move

(2) Two reduced words in S represent the same element $w \in W$ $\Leftrightarrow$ they are related by a finite sequence of Braid moves.

4. Tit's representation

Thm (Tit's): Let I be a finite indexing set, and let $S = \{s_i\}_{i \in I}$, and let $M = \{m_{ij}\}_{i,j \in I}$ be a coxeter matrix. Then there's a faithful representation $\rho : W \to GL_n(\mathbb{R})$, where $W = \langle S \mid (s_i s_j)^{m_{ij}} = e \rangle$, where $n = |S| = |I|$, and st

- $\forall i$, $\rho(s_i) =: \sigma_i$ is a linear involution with fixed point set a hyperplane
- for all i,j, the product $\sigma_i \sigma_j$ has order m_{ij}.

The homo $\rho : W \to GL(n, \mathbb{R})$ is known as the *canonical representation.*

Constructing Tit's rep$^{\underline{n}}$: $|I| = |S| = n$. V have basis $\{e_1, \dots, e_n\}$.

(1) Define sym. bilin. form $B : V \times V \to \mathbb{R}$ by

$$B(e_i, e_j) = \begin{cases} -\cos(\pi / m_{ij}) & \text{if } m_{ij} \text{ finite} \\ -1 & \text{if } m_{ij} \text{ infinite} \end{cases}$$

rem: $B(e_i, e_i) =$, and $B(e_i, e_j) \leq 0$

(2) Define $\sigma_i : V \to V$ by $\sigma_i(v) = v - 2B(e_i, v) e_i$

↳ $\sigma_i(e_i) = -e_i$, $\mathrm{Fix}(\sigma_i) = \{ v \in V : B(e_i, v) = 0 \} =: H_i$

↳ $B(\sigma_i(e_j), \sigma_i(e_k)) = B(e_j, e_k)$

(3) **Prop 4.2:** $\sigma_i \sigma_j$ has order m_{ij} $\forall i,j \in I$

Cor 4.3: $s_i \mapsto \sigma_i$ extends to $\rho : W \to GL_n(\mathbb{R})$

pf: consider cases $\underline{i=j}$: automatic from formula

m_{ij} finite: restrict to V_{ij}, $B|_{V_{ij}}$ is +ve def, so $\sigma_i|$, $\sigma_j|$ look like ortho ref$^{\underline{n}}$s.
Then $\sigma_i \sigma_j| =$ rotation by $\frac{\pi}{m_{ij}}$ $\Rightarrow$ order m_{ij} on V_{ij}. Then note $V = V_{ij} \oplus V_{ij}^{\perp}$,
and $\sigma_i \sigma_j$ acts like id on $V_{ij}^{\perp}$ $\Rightarrow$ order m_{ij} on all V.

$\underline{m_{ij} \text{ infinite:}}$ $B|_{V_{ij}} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ = pos. semidef. $(\sigma_i \sigma_j)^k (e_i) = e_i + 2k(e_i + e_j)$

↳ infinite on V_{ij} $\Rightarrow$ infinite on V.

Faithfulness of Tit's rep$^{\underline{n}}$: Show dual rep$^{\underline{n}}$ $\rho^* : W \to GL(V^*)$ given by
$(\rho^*(w)(\varphi))(w) = \varphi(\rho(w^{-1}))v$ is faithful, ($\Leftrightarrow$ ρ is faithful).

Rem: $\varphi_i \in V^*$ by $\varphi_i = B(e_i, -)$. Then $\rho^*(s_i) := \sigma_i^*$,

$\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$

$H_i^* := \{ \varphi \in V^* \mid \varphi(e_i) = 0 \}$ ← hyperplane of fixed points of σ_i^*

$c_i = \{ \varphi \in V^* \mid \varphi(e_i) > 0 \}$, $c := \bigcap_{i \in I} c_i$, $\bar{c}$ = chamber ass. to rep$^{\underline{n}}$.

Defn: G a group acting on a set Π. $\mathcal{C} \subset \Pi$ is *prefundamental* for G if
$\forall g \in G$, $g\mathcal{C} \cap \mathcal{C} \neq \emptyset \Rightarrow g = 1 \in G$.

↳ c_i prefundamental for $W_i = \langle s_i \rangle$ acting on V^*:

$(s_i \cdot \varphi)(e_i) = \varphi(\rho(s_i^{-1})(e_i)) = \varphi(\sigma_i(e_i)) = \varphi(-e_i) = -\varphi(e_i) < 0$

unless s_i is the identity.

Thm 4.6 (Tits) $\{W, S, \{c_i\}\}$ satisfies property P: $\forall w \in W$, $i \in I$, either $wC \subset C_i$, or $wC \subset s_i C_i$. In the second case, $\ell_S(s_i w) = \ell_S(w) - 1$.

Cor 4.7: C prefundamental for W $\Rightarrow$ ρ^* is faithful.

Change in notation: replace C with $C°$, $\bar{C}$ with C.

Dfn 4.7: Tits cone of (W,S) is $\bigcup_{w \in W} wC \subset V^*$

5. Finite Coxeter Groups

Dfn 5.1: (W,S) Cox. syst. Then (W,S) is *reducible* if $S = S' \sqcup S''$ s.t. $m_{ij} = 2$ $\forall s_i \in S'$, $s_j \in S''$, i.e. $s_i s_j = s_j s_i$ is a rel in W

↳ (W,S) reducible, then $W \cong \langle S' \rangle \times \langle S'' \rangle$

Rem: irreducible can still split as a product: $D_{2(2k)} \cong D_{2k} \times C_2$

Thm 5.3: (W,S) irreducible, $|S| = n$. TFAE:

(1) W a geom. refl^n group on S^{n-1} gen by $S = \{s_i\}_{i \in I}$, and set of refl^{n}s in codim. 1 are faces $\{F_i\}_{i \in I}$ of a simplex in S^{n-1} s.t F_i, F_j meet at $\frac{\pi}{m_{ij}}$.

(2) B is pos. def.

(iii) W is finite

W fin cox. group, $|S| = n$, then $V^* \leftrightarrow \mathbb{E}^n$, and C is a closed Euclidean simplicial cone with boundary given by H_i^*'s.

Cor 4.7 ⇝ C° prefundamental for W $\Leftrightarrow$ $wC^\circ \cap C^\circ \neq \emptyset \Rightarrow w = e$. So if $x \in C^\circ$, then Wx has $|W|$ distinct points

Dfn 5.4: (W,S) finite. *Coxeter polytope* for W is the convex hull of the W orbit of V^* of a point $x \in C^\circ$. (not in general regular)

Dfn 5.6: *Cox-Dynkin diagram* Γ is a simple labelled graph w/ fin. vertex set $V(\Gamma) = S = \{s_i\}_{i \in I}$, edge labels $s_i \overset{m_{ij}}{\text{———}} s_j$, $m_{ij} \geq 3$ or $= \infty$

Lem 5.7: 1-1 correspondence $(W,S) \leftrightarrow$ Cox-Dynk diagrams:

omit $m_{ij} = 3$ label.

$(W,S) \longleftrightarrow \Gamma$ cox. diagram

$S \longleftrightarrow V(\Gamma)$

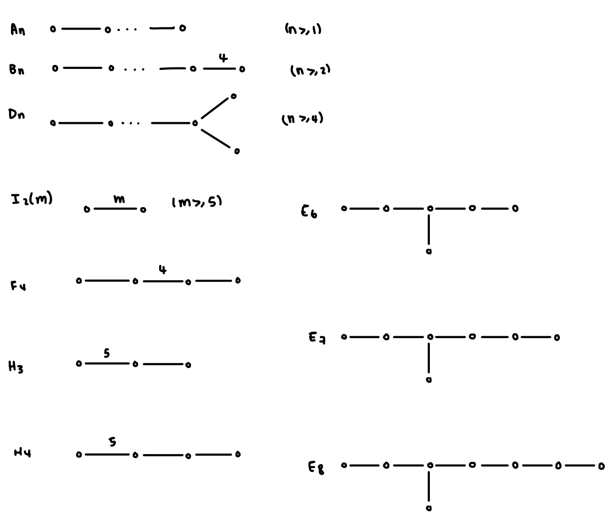
$m_{ij} = m_{ji} = 2 \longleftrightarrow$ no edge between s_i s_j

$m_{ij} \geq 3, = \infty \longleftrightarrow i \overset{m_{ij}}{\text{———}} j$

Denote $\mathrm{Im}(\Gamma)$ by $(W(\Gamma), V(\Gamma))$.

Thm 5.10 (Coxeter) (Classification of finite Coxeter groups)

(W,S) gives fin. Cox. group $\Leftrightarrow$ $(W,S) = (W(\Gamma), V(\Gamma))$ for Γ a disj. $\cup$ of a fin. # of following graphs:



Dfn 5.13: Given a Cox. system (W,S), $T \subseteq S$, then *parabolic subgroup* W_T of W is $:= \langle T \rangle$, $T = \emptyset$, take $W_\emptyset = \{e\}$. Then (W_T, T) a cox system

Lem 5.14: (W,S), then $W(\Gamma_T) \cong W_T$ $\forall$ subsets $T \subseteq S$.

↳ nontrivial proof!

Dfn 5.15: *spherical subgroup* := finite parabolic subgroup

Cor 5.16: all spherical subgroups come from full subgraphs of Γ of (W,S)

Thm 5.18: (W,S) cox. system. Then

(a) (W_T, T) a cox. system $\forall T \subseteq S$

(b) $\forall T \subseteq S$, $w \in W_T$, then $\ell_T(w) = \ell_S(w)$, and any reduced word $w = s_1 \dots s_k$ in S satisfies $s_i \in T$ $\forall i$.

(c) $T, T' \subseteq S$, $W_{T \cap T'} = W_T \cap W_{T'}$, $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$

(d) $T \to W_T$, $\{T \subseteq S\} \to \{\text{parabolic } W_T \leq W\}$ preserves partial ordering by inclusion.

Lem 5.19: (W,S) cox. syst, $w \in W$, then $\exists$ subset $S(w) \subseteq S$ s.t given any reduced word $s_1 \dots s_k$ for w, $S(w) = \{s_1, \dots, s_k\}$.

Dfn 5.20: Cox syst. (W,S), let $\mathcal{S} = \{T \subseteq S : W_T \text{ irreducible}\}$

6. The Basic Construction

Dfn 6.1: **Abstract simplicial complex** is a set V (vertex set), and a collection X of finite subsets of V such that

(1) $\{v\} \in X \ \forall v \in V$ (2) if $\Delta \in X$, and $\Delta' \subset \Delta$, then $\Delta' \in X$

Δ = "abstract simplex". $\Delta' \subsetneq \Delta$, then Δ' a face of Δ

Dfn 6.2: (W,S) cox. sys., X a connected, Hausdorff topological space, a **mirror structure** on X over S is a family of closed, nonempty subsets of X. X is called a **mirrored space** over S, and X_s is the **S-mirror of X**.

Dfn 6.4: (W,S) cox. sys. and X a mirrored space over S, the **nerve** of X $N(X)$ is an abstract simplicial complex with vertex set S and $T \subseteq S$ is a simplex iff $\bigcap_{t \in T} X_t \neq \emptyset$

Examples 6.5:

(1) $X = \text{Cone}\{\sigma_s : s \in S\}$, $X_s = \{\sigma_s\}$, $N(X) = \sqcup\{\sigma_s\}$

(2) Δ^n, $|S| = n+1$. $|S|$ codim-1 faces, labelled by S: $\{\Delta_s : s \in S\}$ $X_s = \Delta_s$

(3) W finite, $V^* \hookrightarrow \mathbb{E}^n$; $\mathcal{C} = \{v \in \mathbb{R}^n : \langle v, e_i \rangle \geq 0 \ \forall i\}$. Then $x \in \mathcal{C}^\circ$, Cox. polytope is orbit Wx. Take $X = \mathcal{C} \cap$ Cox. poly., $X_{s_i} = X \cap H_i$

(W,S) Cox. sys., X mirrored over S, $\exists\, x \in X$ s.t. $x \notin \bigcup_s X_s$. Define $\forall x \in X$ $s(x) = \{s \in S : x \in X_s\}$

Dfn 6.7: W as top. space w/ discrete top, and $W \times X$ with $\times$ top. The **basic construction** is top space w/ quotient top:

$$\mathcal{U}(W,X) = W \times X/\sim$$

where $(w,x) \sim (w',x') \iff x = x'$, and $w^{-1}w' \in W_{s(x)}$ parabolic subgroup $s(x) \subseteq S$.

Equiv. class: $[w,x]$

Rem: $x \in X_s$, then $s \in S(x)$, $\Rightarrow (w,x) \sim (ws,x)$ since $w^{-1}ws = s \in W_{s(x)}$.

Dfn 6.8: $wX := \{w\} \times X$ in $\mathcal{U}(W,X)$. Then wX is a **chamber** of $\mathcal{U}(W,X)$ **Fundamental chamber** eX. wX and wsX are glued along X_s.

Lem 6.9: $X = \text{cone}\{\sigma_s : s \in S\}$, then up to subdivision $\mathcal{U}(W,X) = \text{Cay}_S(W)$

Dfn 6.10: X mirrored space in Exm 6.5 (2), $\mathcal{U}(W,X)$ is called the **Coxeter complex**.

Lemma 6.13: **$\mathcal{U}(W,X)$ a connected** topological space

Dfn 6.14: $\mathcal{U}(W,X)$ is said to be **locally finite** if $\forall\, [w,x] \in \mathcal{U}(W,X)$, $\exists$ an open nhood which meets only finitely many chambers.

Lem 6.16: TFAE:

(1) $\mathcal{U}(W,X)$ locally finite

(2) $\forall x \in X$, $W_{s(x)}$ is finite

(3) $\forall T \subseteq S$ s.t. W_T infinite, then $\bigcap_{t \in T} X_t = \emptyset$

W acts on $\mathcal{U}(W,X)$ by homeos via left action on $W \times X$:

$$w' \cdot (w,x) = (w'w, x)$$

Lem 6.17: Fundamental chamber eX is a strict fund^l domain for $W \curvearrowright \mathcal{U}(W,X)$

$\hookrightarrow \Rightarrow \mathcal{U}(W,X)/W \cong X$.

Bijection: $W \to \mathcal{U}(W,X)$; $w \mapsto wX$

Lem 6.18: **$\text{Stab}_W([w,x])$** $= \{w' \in W : w^{-1}w'w \in W_{s(x)}\}$ (dfn)

$= w W_{s(x)} w^{-1}$.

Lem 6.19: $\mathcal{U}(W,X)$ is Hausdorff

Dfn 6.20: G discrete group, Y hausdorff, then an action by homeos $G \curvearrowright Y$ is **properly discontinuous** if

(i) Y/G is Hausdorff

(ii) $\forall y \in Y$, $G_y = \text{stab}_G(y)$ is finite

(iii) $\forall y \in Y$, $\exists$ open nhood U_y of y in Y s.t. $G_y \cdot U_y = U_y$ (not necessarily pointwise) and $gU_y \cap U_y = \emptyset$ $\forall g \notin G_y$

Lem 6.21: $W \curvearrowright \mathcal{U}(W,X)$ is properly discontinuous iff $W_{s(x)}$ is spherical (finite) $\forall x \in X$.